On some geometric inequalities

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Abstract

In this article we use a purely algebraic inequality to prove a variety of geometric inequalities.

1 Introduction

In the recently published article: An unexpectedly useful inequality by Pham Huu Duc [1], the following inequality was proved

\[(b + c)x + (c + a)y + (a + b)z \geq 2\sqrt{(xy + yz + zx)(ab + bc + ca)} \forall a, b, c, x, y, z \geq 0.\]

The inequality was presented along with its algebraic applications. This inequality not only has many applications in algebra but also it has many applications in geometry. We start with a nice proof of this result that appeared in [2]:

**Proposition 1.** For all real numbers \(a, b, c, x, y, z\) such that \(ab + bc + ca \geq 0\) and \(xy + yz + zx \geq 0\) the following inequality holds

\[(b + c)x + (c + a)y + (a + b)z \geq 2\sqrt{(xy + yz + zx)(ab + bc + ca)}.\]

**Proof.** Using Cauchy-Schwarz inequality we get

\[
(b + c)x + (c + a)y + (a + b)z = (a + b + c)(x + y + z) - (ax + by + cz)
\]

\[
= \sqrt{a^2 + b^2 + c^2 + 2(ab + bc + ca)}[x^2 + y^2 + z^2 + 2(xy + yz + zx)] - (ax + by + cz)
\]

\[
\geq 2\sqrt{(xy + yz + zx)(ab + bc + ca)} + \sqrt{(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)} - (ax + by + cz)
\]

\[
\geq 2\sqrt{(xy + yz + zx)(ab + bc + ca)}.
\]

The next inequality can be proved as a corollary:

**Corollary 1.** For all real positive numbers \(a, b, c, x, y, z\) the following inequality is true

\[
\frac{x}{y + z} a + \frac{y}{z + x} b + \frac{z}{x + y} c \geq \sqrt{3(ab + bc + ca)}.
\]

**Proof.** Let us replace in Proposition 1 \((x, y, z)\) with \(\left(\frac{x}{y+z}, \frac{y}{z+x}, \frac{z}{x+y}\right)\). Note that

\[
\frac{xy}{(z + x)(z + y)} + \frac{yz}{(x + y)(x + z)} + \frac{zx}{(y + z)(y + x)} \geq \frac{3}{4},
\]

and the conclusion follows.
**Proposition 2.** Let $P$ be a point in the plane of triangle $ABC$, then

$$\frac{PA \cdot PC}{bc} + \frac{PC \cdot PA}{ca} + \frac{PA \cdot PB}{ab} \geq 1.$$ 

where $a, b, c$ are the sides of the triangle.

**Proof.** There are many ways to prove this inequality; we use complex numbers. Let the complex coordinates of $A, B, C$ and $P$ be $A(a), B(b), C(c)$ and $P(p)$, respectively.

Using identity


we have

$$BC \cdot PB \cdot PC + CA \cdot PC \cdot PA + AB \cdot PA \cdot PB$$

$$= |(b-c)(p-b)(p-c)| + |(c-a)(p-c)(p-a)| + |(a-b)(p-a)(p-b)|$$

$$\geq |(b-c)(p-b)(p-c) + (c-a)(p-c)(p-a) + (a-b)(p-a)(p-b)|$$

$$= |(a-b)(b-c)(c-a)| = AB \cdot BC \cdot CA.$$ 

Dividing both sides by $AB \cdot BC \cdot CA$ we get

$$\frac{PB \cdot PC}{bc} + \frac{PC \cdot PA}{ca} + \frac{PA \cdot PB}{ab} \geq 1.$$ 

Note that the equality holds if and only if $P = H$, where $H$ is the orthocenter of triangle $ABC$.

Let us combine the ideas of the first two propositions in the following statement:

**Proposition 3.** Let $P$ be a point in the plane of triangle $ABC$, and let $x, y, z$ be real numbers such that $xy + yz + zx \geq 0$. Then

$$(y+z) \frac{PA}{a} + (z+x) \frac{PB}{b} + (x+y) \frac{PC}{c} \geq 2\sqrt{xy + yz + zx}.$$ 

**Proof.** We apply Proposition 1 for $(\frac{PA}{a}, \frac{PB}{b}, \frac{PC}{c})$ and $(x, y, z)$ to get

$$(y+z) \frac{PA}{a} + (z+x) \frac{PB}{b} + (x+y) \frac{PC}{c} \geq 2 \sqrt{(xy + yz + zx) \left( \frac{PA \cdot PC}{bc} + \frac{PC \cdot PA}{ca} + \frac{PA \cdot PB}{ab} \right)}$$

$$\geq 2 \sqrt{xy + yz + zx},$$

as desired.

We continue with a few classical problems that can be solved with the help of the established results.
### 2 Applications

**Problem 1.** Consider triangle $ABC$ and a point $P$ in its plane. Prove that

\[ \frac{PA}{a} + \frac{PB}{b} + \frac{PC}{c} \geq \sqrt{3}. \]

**Solution.** Plugging $x = y = z = 1$ in the Proposition 3, we get

\[ 2 \left( \frac{PA}{a} + \frac{PB}{b} + \frac{PC}{c} \right) \geq 2\sqrt{3}, \]

and we are done.

**Problem 2.** Consider triangle $ABC$ and a point $P$ in its plane. Prove that

\[ a \cdot PA + b \cdot PB + c \cdot PC \geq 4K_{ABC}, \]

where $K_{ABC}$ is the area of triangle $ABC$.

**Solution.** Let $a, b, c$ be the triangle’s sides. Denote \( x = \frac{b^2 + c^2 - a^2}{2}, y = \frac{a^2 + c^2 - b^2}{2}, z = \frac{a^2 + b^2 - c^2}{2}. \) Then

\[ xy + yz + zx = \frac{2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)}{4} = 4K_{ABC}^2 \geq 0. \]

Hence using Proposition 3 for these $(x, y, z)$ we get

\[ a^2 \cdot \frac{PA}{a} + b^2 \cdot \frac{PB}{b} + c^2 \cdot \frac{PC}{c} = a \cdot PA + b \cdot PB + c \cdot PC \geq 4K_{ABC}. \]

**Problem 3.** Let $P$ be a point in the plane of triangle $ABC$. Prove that

\[ PA + PB + PC \geq 6r, \]

where $r$ is the inradius of the incircle of triangle $ABC$.

**Solution.** Let \( x = s - a, y = s - b, z = s - c, \) where $a, b, c$ are the triangle’s sides and $s$ is the semiperimeter. Then using Proposition 3 we get

\[ PA + PB + PC \geq 2\sqrt{(s-a)(s-b) + (s-b)(s-c) + (s-c)(s-a)}. \]

Recall that $(s-a)(s-b) + (s-b)(s-c) + (s-c)(s-a) = r(4R + r)$, where $R$ and $r$ are the circumradius and inradius, respectively. Thus we get a much stronger inequality

\[ PA + PB + PC \geq 2\sqrt{r(4R + r)}. \]
3 References


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